Reduction Using Induced Subnets To Systematically Prove Properties For Free-Choice Nets

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Abstract. We use sequences of t-induced T-nets and p-induced P-nets to convert free-choice nets into T-nets and P-nets while preserving properties such as well-formedness, liveness, lucency, pc-safety, and perpetuality. The approach is general and can be applied to different properties. This allows for more systematic proofs that "peel off" non-trivial parts while retaining the essence of the problem (e.g., lifting properties from T-net and P-net to free-choice nets).

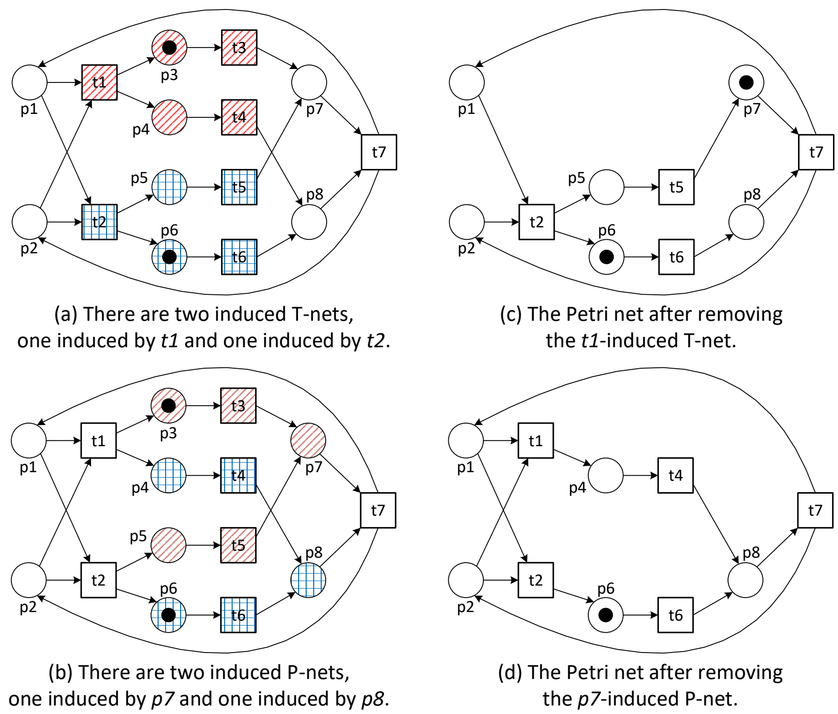
Keywords: Petri Nets · Free-Choice Nets · Net Reduction · Lucency

1 Introduction

Although free-choice nets have been studied extensively, still new and surprising properties are discovered that cannot be proven easily [2]. This paper proposes the use of *T-reductions* and *P-reductions* to prove properties by reducing free-choice nets to either T-nets (marked graphs) or P-nets (state machines). These reductions are based on the notion of *t-induced T-nets* (denoted by $\boxdot_N(t)$) and the notion of *p-induced P-nets* (denoted by $\odot_N(p)$). We propose to use such *reductions* to prove properties that go beyond well-formedness. This paper systematically presents T-reductions and P-reductions, and shows example applications.

Figure 1 illustrates the notion of induced subnets. The original net N has two proper induced T-nets (a) and two proper induced P-nets (b). If the original Petri net N is free-choice and well-formed, then the net after applying the corresponding reduction is still free-choice and well-formed. Think of the original net as an "onion" that is peeled off layer for layer until a T-net or P-net remains. We are interested in *properties that propagate through the different layers*, just like well-formedness. For example, we will show that all perpetual well-formed free-choice nets are lucent, i.e., the existence of a regeneration transition implies that there cannot be two markings enabling the same set of transitions.

The remainder of the paper is organized as follows. Section 2 discusses related work and Section 3 introduces some standard results and notations. Section 4 presents t-induced T-nets and p-induced P-nets and their characteristic properties. The general approach of using T- and P-reductions is presented in Section 5, followed by the application to some properties that go beyond known results like well-formedness (Section 6). Section 7 concludes the paper.



(a) There are two induced T-nets, one induced by *t1* and one induced by *t2*.

(c) The Petri net after removing the *t1*-induced T-net.

(b) There are two induced P-nets, one induced by *p7* and one induced by *p8*.

(d) The Petri net after removing the *p7*-induced P-net.

Fig. 1. A free-choice Petri net N has (a) two proper induced T-nets ($\boxdot_N(t1)$ and $\boxdot_N(t2)$) and (b) two proper induced P-nets ($\odot_N(p7)$ and $\odot_N(p8)$). The Petri nets after removing $\boxdot_N(t1)$ and $\odot_N(p8)$ are shown in (c) and (d).

2 Related Work

For an introduction to free-choice nets and the main known results, we refer to [9, 7]. The work presented in this paper is most related to the completeness proof of the reduction rules in [9]. Proper t-induced T-nets are similar to the CP-nets used in [9]. The use of reduction rules was first proposed and studied by Berthelot [6]. Desel provided reduction rules for free-choice nets without frozen tokens [8]. Indirectly related are the blocking theorem [12, 15] and the notion of lucency in perpetual free-choice nets [2, 4]. To get a deeper understanding of well-formed free-choice nets, we also refer to [14, 11]. The problem addressed in this paper was inspired by questions originating from the process mining domain [1], e.g., see [10] for the application of traditional reduction rules in process discovery and see [4] for the relation between lucency and translucent event logs.

3 Preliminaries

This section introduces basic mathematical concepts and some well-known Petri net notions and results.

$\mathcal{B}(A)$ is the set of all *multisets* over some set A, e.g., $b = [x^3, y^2, z] \in \mathcal{B}(A)$ is a multiset with 6 elements ($|B| = 6$). We assume the standard multiset operators $\in$ (element), $\uplus$ (union), $\setminus$ (difference), $\leq$ (smaller or equal), and $<$ (smaller). $\sigma = \langle a_1, a_2, \ldots, a_n\rangle \in X^*$ denotes a *sequence* over X of length $|\sigma| = n$. $\sigma_i = a_i$ for $1 \leq i \leq |\sigma|$. $\langle\, \rangle$ is the empty sequence.

Definition 1 (Petri Net). *A Petri net is a tuple $N = (P, T, F)$ with P the set of places, T the set of transitions such that $P \cap T = \emptyset$, and $F \subseteq (P \times T) \cup (T \times P)$ the flow relation such that the graph $(P \cup T, F)$ is non-empty and weakly connected. A Petri net is non-trivial if $F \neq \emptyset$ (i.e., there is at least one place and one transition).*

Definition 2 (Marking). *Let $N = (P, T, F)$ be a non-trivial Petri net. A marking M is a multiset of places, i.e., $M \in \mathcal{B}(P)$. (N, M) is a marked net.*

The requirement that a marked net is non-trivial (i.e., $F \neq \emptyset$), together with the requirement that $(P \cup T, F)$ is weakly connected is there to avoid uninteresting border cases (nets without places or transitions cannot change state and unconnected parts can be analyzed separately). For a subset of places $X \subseteq P$: $M{\restriction_X} = [p \in M \mid p \in X]$ is the marking *projected* on this subset. $M(X) = \sum_{p\in X} M(p) = |M{\restriction_X}|$ is the total number of tokens in X.

A Petri net $N = (P, T, F)$ defines a directed graph with nodes $P \cup T$ and edges F. For any $x \in P \cup T$, $\bullet x = \{y \mid (y, x) \in F\}$ denotes the set of input nodes and $x\bullet = \{y \mid (x, y) \in F\}$ denotes the set of output nodes. The notation can be generalized to sets: $\bullet X = \{y \mid \exists_{x\in X}\ (y, x) \in F\}$ and $X\bullet = \{y \mid \exists_{x\in X}\ (x, y) \in F\}$ for any $X \subseteq P \cup T$.

Definition 3 (Elementary Paths and Circuits). *A* path *in a Petri net $N = (P, T, F)$ is a non-empty ($n \geq 1$) sequence of nodes $\rho = \langle x_1, x_2, \ldots, x_n\rangle$ such that $(x_i, x_{i+1}) \in F$ for $1 \leq i < n$. $paths(N) \subseteq (P \cup T)^*$ is the set of all paths in N. ρ is an* elementary path *if $x_i \neq x_j$ for $1 \leq i < j \leq n$ (i.e., no element occurs more than once). An elementary path is called a* circuit *if $(x_n, x_1) \in F$.*

A transition $t \in T$ is *enabled* in marking M of net N, denoted as $(N, M)[t\rangle$, if each of its input places $\bullet t$ contains at least one token. $en(N, M) = \{t \in T \mid (N, M)[t\rangle\}$ is the set of enabled transitions.

An enabled transition t may *fire*, i.e., one token is removed from each of the input places $\bullet t$ and one token is produced for each of the output places $t\bullet$. Formally: $M' = (M \setminus \bullet t) \uplus t\bullet$ is the marking resulting from firing enabled transition t in marking M of Petri net N. $(N, M)[t\rangle(N, M')$ denotes that t is enabled in M and firing t results in marking M'.

Let $\sigma = \langle t_1, t_2, \ldots, t_n\rangle \in T^*$ be a sequence of transitions ($n \geq 0$). $(N, M)[\sigma\rangle (N, M')$ denotes that there is a set of markings $M_1, M_2, \ldots, M_{n+1}$ ($n \geq 0$) such that $M_1 = M$, $M_{n+1} = M'$, and $(N, M_i)[t_i\rangle(N, M_{i+1})$ for $1 \leq i \leq n$. A marking M' is *reachable* from M if there exists a *firing sequence* σ such that $(N, M)[\sigma\rangle(N, M')$. $R(N, M) = \{M' \in \mathcal{B}(P) \mid \exists_{\sigma\in T^*}\ (N, M)[\sigma\rangle(N, M')\}$ is the set of all reachable markings.

Definition 4 (Live, Bounded, Safe, Dead, Deadlock-free, Well-Formed). *A marked net (N, M) is* live *if for every reachable marking $M' \in R(N, M)$ and every transition $t \in T$ there exists a marking $M'' \in R(N, M')$ that enables t. A marked net (N, M) is k-bounded if for every reachable marking $M' \in R(N, M)$ and every $p \in P$: $M'(p) \leq k$. A marked net (N, M) is* bounded *if there exists a k such that (N, M) is k-bounded. A 1-bounded marked net is called* safe. *A place $p \in P$ is* dead *in (N, M) when it can never be marked (no reachable marking marks p). A transition $t \in T$ is* dead *in (N, M) when it can never be enabled (no reachable marking enables t). A marked net (N, M) is* deadlock-free *if each reachable marking enables at least one transition. A Petri net N is* structurally bounded *if (N, M) is bounded for any marking M. A Petri net N is* structurally live *if there exists a marking M such that (N, M) is live. A Petri net N is* well-formed *if there exists a marking M such that (N, M) is live and bounded.*

For particular subclasses of Petri nets, there are various relationships between structural properties and behavioral properties like liveness and boundedness [7]. In this paper, we focus on free-choice nets [9].

Definition 5 (P-net, T-net, and Free-choice Net). *Let $N = (P, T, F)$ be a Petri net. N is a* P-net *(also called a state machine) if $|\bullet t| = |t\bullet| = 1$ for any $t \in T$. N is a* T-net *(also called a marked graph) if $|\bullet p| = |p\bullet| = 1$ for any $p \in P$. N is a* free-choice net *if for any $t_1, t_2 \in T$: $\bullet t_1 = \bullet t_2$ or $\bullet t_1 \cap \bullet t_2 = \emptyset$. N is* strongly connected *if the graph $(P \cup T, F)$ is strongly connected, i.e., for any two nodes x and y there is a path leading from x to y.*

Definition 6 (Cluster). *Let $N = (P, T, F)$ be a Petri net and $x \in P \cup T$. The* cluster *of node x, denoted $[x]_c$ is the smallest set such that (1) $x \in [x]_c$, (2) if $p \in [x]_c \cap P$, then $p\bullet \subseteq [x]_c$, and (3) if $t \in [x]_c \cap T$, then $\bullet t \subseteq [x]_c$.*

Definition 7 (Subnet, Complement, P-component, T-Component). *Let $N = (P, T, F)$ be a Petri net and $X \subseteq P \cup T$. $N\restriction_X = (P \cap X, T \cap X, F \cap (X \times X))$ is the* subnet *generated by X. $N \setminus\!\!\setminus X = (P \setminus X, T \setminus X, F \cap (((P \cup T) \setminus X) \times ((P \cup T) \setminus X)))$ is the* complement *generated by X. $N\restriction_X$ is a* P-component *of N if $\bullet p \cup p\bullet \subseteq X$ for $p \in X \cap P$ and $N\restriction_X$ is a strongly connected P-net. $N\restriction_X$ is a* T-component *of N if $\bullet t \cup t\bullet \subseteq X$ for $t \in X \cap T$ and $N\restriction_X$ is a strongly connected T-net. $PComp(N) = \{X \subseteq P \cup T \mid N\restriction_X$ is a P-component$\}$. $TComp(N) = \{X \subseteq P \cup T \mid N\restriction_X$ is a T-component$\}$.*

Definition 8 (P-cover, T-cover). *Let $N = (P, T, F)$ be a Petri net. N has a* P-cover *if $\bigcup PComp(N) = P \cup T$.[3] N has a* T-cover *if $\bigcup TComp(N) = P \cup T$.*

Theorem 1 (Coverability Theorems [9]). *Let $N = (P, T, F)$ be a well-formed free-choice net. $\bigcup PComp(N) = \bigcup TComp(N) = P \cup T$.*

Moreover, for any well-formed free-choice net N and marking M: (N, M) is live if and only if every P-component is marked in M (Theorem 5.8 in [9]).

The dual Petri net is the net where the role of places and transitions is swapped and the arcs are reversed.

[3] $\bigcup Q = \bigcup_{X \in Q} X$ for some set of sets Q.

Definition 9 (Dual Net). *Let $N = (P, T, F)$ be a Petri net. $N^{dual} = (T, P, F^{-1})$ with $F^{-1} = \{(x, y) \mid (y, x) \in F\}$ is the dual net of N.*

Note that $(N^{dual})^{dual} = N$. We also use the following well-known result [9, 13].

Theorem 2 (Duality Theorem). *Let N be a Petri net and N^{dual} the dual net of N. N is a well-formed free-choice net if and only if N^{dual} is a well-formed free-choice net.*

4 Induced Subnets in Free-Choice Nets: Existence and Properties

We start by introducing the notion of t-induced T-nets, i.e., subnets fully defined by an initial transition t and all nodes that can be reached from t without visiting places with multiple input or multiple output transitions. Figure 1 highlights two induced T-nets: $\boxdot_N(t1) = \{t1, p3, p4, t3, t4\}$ and $\boxdot_N(t1) = \{t2, p5, p6, t5, t6\}$.

Definition 10 (t-Induced T-net). *Let $N = (P, T, F)$ be a Petri net and $t \in T$. $\boxdot_N(t) \subseteq P \cup T$ is the smallest set such that*

- *$t \in \boxdot_N(t)$,*
- *$\{p' \in t'\bullet \mid |\bullet p'| = 1 \ \wedge \ |p'\bullet| = 1\} \subseteq \boxdot_N(t)$ for any $t' \in \boxdot_N(t) \cap T$, and*
- *$p'\bullet \subseteq \boxdot_N(t)$ for any $p' \in \boxdot_N(t) \cap P$.*

$\boxdot_N(t)$ are the nodes of the t-induced T-net of N that is denoted by $N_{\boxdot(t)} = N{\restriction}_{\boxdot_N(t)}$. $\overline{N_{\boxdot(t)}} = N \setminus\!\!\setminus \boxdot_N(t)$ is the complement of the t-induced T-net of N. $\boxdot_N(t)$ is proper if the complement $\overline{N_{\boxdot(t)}}$ is a non-trivial strongly-connected Petri net.

Informally, a t-induced T-net can be viewed as the union of a set of elementary paths that all start in t and have non-branched places. A t-induced T-net is proper if after removal the net is strongly-connected. $\boxdot_N(t1)$ is a *proper* $t1$-induced T-net of the net N in Figure 1(a), because removing all the nodes in $\boxdot_N(t1)$ leaves the strongly-connected Petri net $\overline{N_{\boxdot(t1)}}$ depicted in Figure 1(c). Proper t-induced T-nets have the following properties.

Proposition 1 (Properties of Proper t-Induced T-net). *Let $N = (P, T, F)$ be a strongly-connected free-choice net and $\boxdot_N(t)$ a proper t-induced T-net of N.*

(1) $N_{\boxdot(t)}$ is a T-net.
(2) $\overline{N_{\boxdot(t)}}$ is free-choice.
(3) For all $p' \in \boxdot_N(t) \cap P$: $\bullet p' \cup p'\bullet \subseteq \boxdot_N(t)$.
(4) For all $t' \in \boxdot_N(t) \cap (T \setminus \{t\})$: $\bullet t' \subseteq \boxdot_N(t)$.
(5) $\bullet t \subseteq P \setminus \boxdot_N(t)$.
(6) There is a $t' \in T \setminus \boxdot_N(t)$ such that $\bullet t = \bullet t'$.
(7) For any path $\rho = \langle x_1, x_2, \ldots, x_n \rangle \in paths(N)$ such that $x_1 \notin \boxdot_N(t)$ and $x_n \in \boxdot_N(t)$: $t \in \{x_2, \ldots, x_n\}$.
(8) For any proper t'-induced T-net of N: $t' = t$ or $\boxdot_N(t') \cap \boxdot_N(t) = \emptyset$.

Proof. (1) $N_{\boxdot(t)}$ is a T-net, because, by construction, all added places have one input transition and one output transition, and only nodes connected to other nodes are added. (2) Removing a node and all connected arcs cannot invalidate the free-choice property.

The connections between the remaining places and transitions do not change.
(3) The t-induced T-net is transition bordered, i.e., for each place in $\boxdot_N(t)$ the unique input transition and output transition are added.
(4) If $t' \in \boxdot_N(t) \cap (T \setminus \{t\})$, then there is at least one input place $p' \in \bullet t' \cap \boxdot_N(t)$ (by construction, transitions different from t are only added to $\boxdot_N(t)$ after an input place was added). Assume t' has an input place outside $\boxdot_N(t)$, i.e., $p'' \in \bullet t' \setminus \boxdot_N(t)$. Since $\overline{N_{\boxdot(t)}}$ is strongly-connected, there must be a $t'' \in p'' \bullet \setminus \boxdot_N (t)$ (otherwise p' would be a sink place in $\overline{N_{\boxdot(t)}}$). Since the net is free-choice, $\bullet t' = \bullet t''$ and $p' \in \bullet t''$. This contradicts with (3).
(5) Since N is strongly connected, there must be an arc from a node outside $\boxdot_N(t)$ to a node inside $\boxdot_N(t)$. Using (3) and (4), the node inside $\boxdot_N(t)$ must be t. Hence, there is a place $p' \in \bullet t \setminus \boxdot_N(t)$. Since $\overline{N_{\boxdot(t)}}$ is strongly-connected, there must be a $t' \in p' \bullet \setminus \boxdot_N (t)$ (otherwise p' would be a sink place in $\overline{N_{\boxdot(t)}}$). Since the net is free-choice, $\bullet t = \bullet t'$. Assume that t has an input place inside $\boxdot_N(t)$, then also t' has an input place inside $\boxdot_N(t)$. This leads to a contradiction because the t-induced T-net is transition bordered. Hence, t cannot have an input place inside $\boxdot_N(t)$, i.e., $\bullet t \subseteq T \setminus \boxdot_N(t)$.
(6) The input places of t remain after removing $\boxdot_N(t)$ and the complement is strongly connected. Hence, there must be a transition $t' \in T \setminus \boxdot_N(t)$ such that $\bullet t = \bullet t'$.
(7) The only way to "enter" the t-induced T-net is through t. This directly follows from (3), (4), and (5).
(8) Assume $x \in \boxdot_N(t) \cap \boxdot_N(t')$. This implies that there must be an elementary non-branched path (i.e., places on the path have one input and one output transition) from t' to x. Using (3) and (4), we can follow this path backwards from x to t' and conclude that all the nodes belong to $\boxdot_N(t)$, including t', i.e., $t' \in \boxdot_N(t)$. Applying (4) once more assuming $t \neq t'$ shows that $\bullet t' \subseteq \boxdot_N(t)$. However, using (6) we know that t' is involved in a choice, making the input places branching and thus leading to a contradiction. $\square$

A p-induced P-net is a subnet fully defined by a final place p and all nodes from which p can be reached without visiting transitions with multiple input or multiple output places. Figure 1 highlights two induced P-nets: $\odot_N(p7) = \{p7, t3, t5, p3, p5\}$ and $\odot_N(p8) = \{p8, t4, t6, p4, p6\}$.

Definition 11 (p-Induced P-net). *Let $N = (P, T, F)$ be a Petri net and $p \in P$. $\odot_N(p) \subseteq P \cup T$ is the smallest set such that*

- $p \in \odot_N(p)$,
- $\{t' \in \bullet p' \mid |\bullet t'| = 1 \ \wedge\ |t' \bullet| = 1\} \subseteq \odot_N(p)$ *for any* $p' \in \odot_N(p) \cap P$, *and*
- $\bullet t' \subseteq \odot_N(p)$ *for any* $t' \in \odot_N(p) \cap T$.

$\odot_N(p)$ *are the nodes of the p-induced P-net of N which is denoted by* $N_{\odot(p)} = N\restriction_{\odot_N(p)}$. $\overline{N_{\odot(p)}} = N \setminus\!\!\setminus \odot_N(p)$ *is the complement of the p-induced P-net of N. $\odot_N(p)$ is proper if the complement $\overline{N_{\odot(p)}}$ is a non-trivial strongly-connected Petri net.*

Due to duality, symmetric properties can be found using similar reasoning.

Proposition 2 (Properties of Proper p-Induced P-net). *Let $N = (P, T, F)$ be a strongly-connected free-choice Petri net and $\odot_N(p)$ a proper p-induced P-net of N.*

(1) $N_{\odot(p)}$ *is a P-net.*
(2) $\overline{N_{\odot(p)}}$ *is free-choice.*
(3) *For all* $t' \in \odot_N(p) \cap T$*:* $\bullet t' \cup t'\bullet \subseteq \odot_N(p)$*.*
(4) *For all* $p' \in \odot_N(p) \cap (P \setminus \{p\})$*:* $p'\bullet \subseteq \odot_N(p)$*.*
(5) $p\bullet \subseteq T \setminus \odot_N(p)$*.*
(6) *There is a* $p' \in P \setminus \odot_N(p)$ *such that* $p\bullet = p'\bullet$*.*
(7) *For any path* $\rho = \langle x_1, x_2, \ldots, x_n \rangle \in \mathit{paths}(N)$ *such that* $x_1 \in \odot_N(p)$ *and* $x_n \notin \odot_N(p)$*:* $p \in \{x_1, x_2, \ldots, x_{n-1}\}$*.*
(8) *For any proper* p'*-induced P-net of* N*:* $p' = p$ *or* $\odot_N(p') \cap \odot_N(p) = \emptyset$*.*

Proof. Analogous to Proposition 1. □

Next, we show that a t-induced T-net corresponds to a t-induced P-net in the dual net (where t is a place). Recall that places and transitions are exchanged and the direction of all arcs is reversed in N^{dual}.

Lemma 1 (Duality Lemma for Induced Subnets). *Let* $N = (P, T, F)$ *be a Petri net.*
(1) *For any* $t \in T$*:* $\boxdot_N(t)$ *is a (proper)* t*-induced T-net of* N *if and only if* $\odot_{N^{dual}}(t)$ *is a (proper)* t*-induced P-net of* N^{dual}*.*
(2) *For any* $p \in P$*:* $\odot_N(p)$ *is a (proper)* p*-induced P-net of* N *if and only if* $\boxdot_{N^{dual}}(p)$ *is a (proper)* p*-induced T-net of* N^{dual}*.*

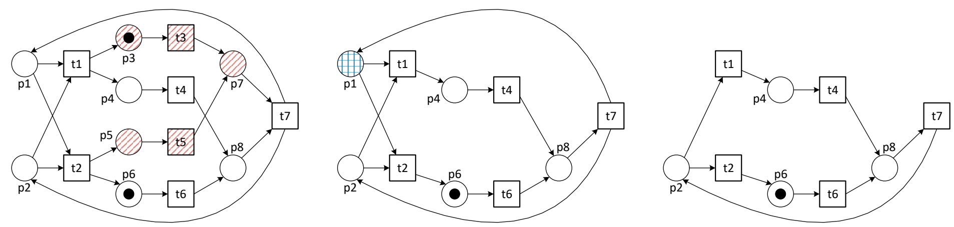
Proof. Let $N = (P, T, F)$ be a Petri net and $N' = (P', T', F') = N^{dual} = (T, P, F^{-1})$ the dual net. $\bullet$ is used for the pre and post sets in N and $\circ$ is used for the pre and post sets in N'. Note that $x \in \bullet y \Leftrightarrow (x, y) \in F \Leftrightarrow (y, x) \in F' \Leftrightarrow x \in y\circ$. Similarly, $x \in y\bullet \Leftrightarrow (y, x) \in F \Leftrightarrow (x, y) \in F' \Leftrightarrow x \in \circ y$. Using these insights and a pairwise comparison of the three rules in Definition 10 and Definition 11, the proof follows immediately. □

Proposition 3 (Induced Subnets Relate to T/P-Components). *Let* $N = (P, T, F)$ *be a well-formed free-choice net. For any proper* t*-induced T-net* $\boxdot_N(t)$*, there exists a T-component* $X \in \mathit{TComp}(N)$ *such that* $\boxdot_N(t) \subseteq X$*. For any proper* p*-induced P-net* $\odot_N(p)$*, there exists a P-component* $X \in \mathit{PComp}(N)$ *such that* $\odot_N(p) \subseteq X$*.*

Proof. Let $\boxdot_N(t)$ be a proper t-induced T-net. t must be covered by some T-component X (Theorem 1). If t is included, then also $t\bullet$ is included in X. For the places in $t\bullet$ that have only one output transition, also these output transitions need to be included in X. For all transitions included, the input and output places must be included in X. Etc. Hence, $\boxdot_N(t) \subseteq X$. Let $\odot_N(p)$ be a proper p-induced P-net. We can apply the same reasoning since p is covered by some P-component X (Theorem 1). However, now the arcs are followed in the reverse direction to show that $\odot_N(p) \subseteq X$. □

Next, we show that a well-formed free-choice net has at least two induced T-nets or is a T-net. The proof combines Proposition 7.11 in [9] with Lemma 1.2 in [15].

Lemma 2 (Existence of t-Induced T-nets). *Let* $N = (P, T, F)$ *be a well-formed free-choice net.* N *is either a T-net or there exist at least two different transitions* $t_1, t_2 \in T$ *such that* $\boxdot_N(t_1)$ *is proper and* $\boxdot_N(t_2)$ *is proper.*



(a) Original Petri net.

(b) The Petri net after removing the *p7*-induced P-net.

(c) The Petri net after removing the *p1*-induced P-net.

Fig. 2. A well-formed free choice net is reduced in two steps into a P-net using $\gamma = \langle p7, p1 \rangle$.

Proof. N is covered by the set of T-components $TComp(N)$. Take a minimal $Q \subseteq TComp(N)$ such that $\bigcup Q = P \cup T$ (i.e., removing a T-component from Q leads to incomplete coverage of the net). Assume $|Q| \geq 2$ (otherwise N is a T-net). Create a spanning tree for the graph $G = (V, E)$ with $V = Q$ and $E = \{(X_1, X_2) \in Q \times Q \mid X_1 \cap X_2 \neq \emptyset\}$. Pick a T-component $X \in Q$ that is a leaf in the spanning tree (i.e., the remaining T-components in $Q' = Q \setminus \{X\}$ are still connected). There are at least two such leaf nodes, because $|Q| \geq 2$. Let $Y = X \setminus \bigcup Q'$ be the nodes only in X. $Y \neq \emptyset$ because Q was minimal. Let $Y' \subseteq Y$ be a maximal connected subset of Y. Obviously, Y' is a transition-bordered connected T-net. $N \setminus\!\!\setminus Y$ is strongly-connected because the remaining T-components in $Q' = Q \setminus \{X\}$ are still connected and inside a T-component all nodes are strongly-connected. The nodes in $Y \setminus Y'$ are not connected to Y' (due to maximality) and therefore connected to nodes in $\bigcup Q'$. Hence, $N \setminus\!\!\setminus Y'$ is strongly-connected. Moreover, there can only be one transition in Y' consuming tokens from $\bigcup Q'$. This implies that Y' corresponds to a proper induced T-net. (Note that we use the same reasoning as in Def. 7.7, Prop. 7.10, and Prop. 7.11 in [9].)

We could have picked two different T-components $X_1, X_2 \in Q$ that are leaves in the spanning tree. Therefore, it is possible to find at least two different connected subsets that are non-overlapping. Hence, we can find two transitions $t_1, t_2 \in T$ such that both $\boxdot_N(t_1)$ and $\boxdot_N(t_2)$ are proper. □

We can use a similar approach to show that a well-formed free-choice net has at least two induced P-nets or is a P-net. Consider the well-formed free-choice net in Figure 2(a). This is not a P-net, so we can find at least two induced P-nets: $\odot_N(p7)$ and $\odot_N(p8)$. After removing the nodes in $\odot_N(p7)$, we get $N' = \overline{N_{\odot(p7)}} = N \setminus\!\!\setminus \odot_N(p7)$ shown in Figure 2(b). In N' there are again at least two induced P-nets $\odot_{N'}(p1)$ and $\odot_{N'}(p2)$. After removing the nodes in $\odot_{N'}(p1)$, we obtain the P-net $\overline{N'_{\odot(p1)}}$ shown in Figure 2(c).

Lemma 3 (Existence of p-Induced P-nets). *Let $N = (P, T, F)$ be a well-formed free-choice net. N is either a P-net or there exist at least two different places $p_1, p_2 \in P$ such that $\odot_N(p_1)$ is proper and $\odot_N(p_2)$ is proper.*

Proof. Let $N = (P, T, F)$ be a well-formed free-choice net. $N' = (P', T', F') = N^{dual} = (T, P, F^{-1})$ is the dual net. N' is also a well-formed free-choice net (Theorem 2). N is a P-net if and only if N' is a T-net. If N is not a P-net, then N' is not a

T-net and there exists a $t \in T' = P$ such that $\boxdot_{N'}(t)$ is a proper t-induced T-net of N' (apply Lemma 2). Using Lemma 1, this implies that $\odot_N(t)$ is a proper t-induced P-net of N for some place $t \in P$. A similar reasoning can be used to show that there are at least two proper induced P-nets $\odot_N(p_1)$ and $\odot_N(p_2)$. □

Thus far, we ignored the marking of the free-choice net when removing an induced T-net or P-net. Removing an induced T-net and its tokens may destroy liveness. In Figure 1, we had to "push out" the token in $p3$ to $p7$ to preserve liveness.

Proposition 4 (Pushed Out Markings Exist And Are Unique). *Let (N, M) be a strongly-connected marked free-choice net, $\boxdot_N(t)$ a proper t-induced T-net of N, $\hat{T} = (\boxdot_N(t) \cap T) \setminus \{t\}$, and $push(\boxdot_N(t), M) = \{M' \in \mathcal{B}(P) \mid \exists_{\sigma \in (\hat{T})^*} (N, M)[\sigma\rangle (N, M') \wedge en(N, M') \cap \hat{T} = \emptyset\}$. $|push(\boxdot_N(t), M)| = 1$.*

Proof. Follows directly from the properties listed in Proposition 1. For each transition $t' \in \hat{T} = (\boxdot_N(t) \cap T) \setminus \{t\}$, there is an elementary path from t to t' where each place has one input transition and one output transition. Since only transitions in $\hat{T}$ are considered in $push(\boxdot_N(t), M)$, the number of tokens on a path cannot increase, but decreases when t' fires. This applies to any $t' \in \hat{T}$, hence, after some time none of the transitions in $\hat{T}$ can fire anymore and we find a marking M' such that $en(N, M') \cap \hat{T} = \emptyset$. Since $N_{\boxdot(t)}$ is a T-net, all interleavings lead to the same M'. □

Since the "pushed out marking" is unique, we can update the marking after removing a t-induced T-net in a deterministic manner. When a p-induced P-net is removed, we can simply project the marking onto the remaining places.

Definition 12 (Updated Markings). *Let (N, M) be a marked Petri net, $N = (P, T, F)$, $\boxdot_N(t)$ a proper t-induced T-net of N, and $\odot_N(p)$ a proper p-induced P-net of N.*

- *$mrk_{\boxdot}(N, t, M) \in \{M' \restriction_{P \setminus \boxdot_N(t)} \mid M' \in push(\boxdot_N(t), M)\}$ is the unique marking obtained by "pushing out" tokens as much as possible (see Proposition 4).*
- *$mrk_{\odot}(N, p, M) = M \restriction_{P \setminus \odot_N(p)}$ is the unique marking obtained by removing the tokens in $\odot_N(p)$.*

Lemma 4 (Well-Formedness of $\overline{N_{\boxdot(t)}}$). *Let $N = (P, T, F)$ be a well-formed free-choice net having a transition $t \in T$ such that $\boxdot_N(t)$ is proper. $\overline{N_{\boxdot(t)}} = (\overline{P}, \overline{T}, \overline{F})$ is the corresponding complement.*

(1) For any $\overline{M}, \overline{M}' \in \mathcal{B}(\overline{P})$, $\hat{M} \in \mathcal{B}(P)$, and $\sigma \in \overline{T}^*$: if $(\overline{N_{\boxdot(t)}}, \overline{M})[\sigma\rangle(\overline{N_{\boxdot(t)}}, \overline{M}')$, then $(N, \overline{M} \uplus \hat{M})[\sigma\rangle(N, \overline{M}' \uplus \hat{M})$.

(2) For any $M \in \mathcal{B}(P)$: if (N, M) is live and bounded, then $(\overline{N_{\boxdot(t)}}, mrk_{\boxdot}(N, t, M))$ is live and bounded.

(3) $\overline{N_{\boxdot(t)}}$ is well-formed and free-choice.

Proof. Let $N = (P, T, F)$ be a well-formed free-choice net, $\boxdot_N(t)$ a proper t-induced T-net, and $\overline{N_{\boxdot(t)}} = (\overline{P}, \overline{T}, \overline{F})$. $\overline{N_{\boxdot(t)}}$ is free-choice (apply Proposition 1(2)).

(1) If $(\overline{N_{\boxdot(t)}}, \overline{M})[\sigma\rangle(\overline{N_{\boxdot(t)}}, \overline{M}')$, then $(N, \overline{M})[\sigma\rangle(N, \overline{M}')$ (because $\overline{T} \subseteq T$ and $\bullet t$ and $t\bullet$ are the same for $t \in \overline{T}$ in both nets). Adding tokens cannot disable an enabled firing sequence. Hence, $(N, \overline{M} \uplus \hat{M})[\sigma\rangle(N, \overline{M}' \uplus \hat{M})$.

(2) Assume (N, M) is live and bounded. $M' \in push(\boxdot_N(t), M)$ is the unique "pushed out marking" (see Proposition 4). Obviously, (N, M') is also live and bounded. Split M' into $\overline{M} = mrk_{\boxdot}(N, t, M) = M' \restriction_{P \setminus \boxdot_N(t)}$ and $\hat{M} = M' \setminus mrk_{\boxdot}(N, t, M)$, i.e., $M' = \overline{M} \uplus \hat{M}$. We need to show that $(\overline{N_{\boxdot(t)}}, \overline{M})$ is live and bounded.

Using (1) we know that any firing sequence enabled in $(\overline{N_{\boxdot(t)}}, \overline{M})$ is also enabled in $(N, \overline{M} \uplus \hat{M})$. Hence, $(\overline{N_{\boxdot(t)}}, \overline{M})$ is bounded, because $(N, \overline{M} \uplus \hat{M})$ is bounded.

$(\overline{N_{\boxdot(t)}}, \overline{M})$ is a bounded, strongly-connected, and free-choice. Using Theorem 4.31 in [9], we know that $(\overline{N_{\boxdot(t)}}, \overline{M})$ is live if and only if $(\overline{N_{\boxdot(t)}}, \overline{M})$ is deadlock-free. Assume $(\overline{N_{\boxdot(t)}}, \overline{M})$ has a reachable deadlock $\overline{M}_D$. The corresponding reachable marking from (N, M') is $M_D = \overline{M}_D \uplus \hat{M}$ (recall $M' = \overline{M} \uplus \hat{M}$). The transitions in $\overline{T} \cup \{t\}$ are also disabled in (N, M_D) because the input places are unaffected (note that there is a $t' \in \overline{T}$ such that $\bullet t = \bullet t'$ that is disabled and so is t). The other transitions in $\hat{T} = (\boxdot_N(t) \cap T) \setminus \{t\}$ are also dead because we started from a marking where tokens were "pushed out" until no transition in $\hat{T}$ was enabled anymore. Hence, also M_D is a dead reachable marking contradicting that (N, M') is live. Hence, $(\overline{N_{\boxdot(t)}}, \overline{M})$ cannot have a reachable deadlock, implying that $(\overline{N_{\boxdot(t)}}, \overline{M})$ is live.

(3) Because N is well-formed there is a marking M such that (N, M) is live and bounded. $\overline{N_{\boxdot(t)}}$ is well-formed because $(\overline{N_{\boxdot(t)}}, mrk_{\boxdot}(N, t, M))$ is live and bounded (follows directly from (2)). $\square$

We can also show that removing a p-induced P-net does not jeopardize liveness and boundedness. Note that $mrk_{\odot}(N, p, M)$ is obtained by simply removing the tokens in $\odot_N(p)$ (Definition 12).

Lemma 5 (Well-Formedness of $\overline{N_{\odot(p)}}$). *Let $N = (P, T, F)$ be a well-formed free-choice net having a place $p \in P$ such that $\odot_N(p)$ is proper. $\overline{N_{\odot(p)}} = (\overline{P}, \overline{T}, \overline{F})$ is the corresponding complement.*

(1) For any $M, M' \in \mathcal{B}(P)$ and $\sigma \in T^*$: if $(N, M)[\sigma\rangle(N, M')$, then $(\overline{N_{\odot(p)}}, mrk_{\odot}(N, p, M))[\sigma\restriction_{\overline{T}}\rangle(\overline{N_{\odot(p)}}, mrk_{\odot}(N, p, M'))$.

(2) $\overline{N_{\odot(p)}}$ is well-formed and free-choice.

(3) For any $M \in \mathcal{B}(P)$: if (N, M) is live and bounded, then $(\overline{N_{\odot(p)}}, mrk_{\odot}(N, p, M))$ is live and bounded.

Proof. Let $N = (P, T, F)$ be a well-formed free-choice net, $\odot_N(p)$ a proper p-induced P-net, and $\overline{N_{\odot(p)}} = (\overline{P}, \overline{T}, \overline{F})$. Recall that $mrk_{\odot}(N, p, M) = M\restriction_{P \setminus \odot_N(p)} = M\restriction_{\overline{P}}$ and $mrk_{\odot}(N, p, M') = M'\restriction_{\overline{P}}$. In the proof, we use these more compact notations.

(1) If $(N, M)[t\rangle(N, M')$ and $t \in \overline{T}$, then $(\overline{N_{\odot(p)}}, M\restriction_{\overline{P}})[t\rangle(\overline{N_{\odot(p)}}, M'\restriction_{\overline{P}})$ because removing places cannot disable a transition. If $(N, M)[t\rangle(N, M')$ and $t \notin \overline{T}$, then we can ignore t, because t is not impacting places in $\overline{P}$ and $M\restriction_{\overline{P}} = M'\restriction_{\overline{P}}$. Iteration over all transitions in σ shows that indeed $(\overline{N_{\odot(p)}}, M\restriction_{\overline{P}})[\sigma\restriction_{\overline{T}}\rangle(\overline{N_{\odot(p)}}, M'\restriction_{\overline{P}})$.

(2) Since $N = (P, T, F)$ is a well-formed free-choice net, $N^{dual} = (T, P, F^{-1})$ is a well-formed free-choice net (apply Theorem 2). Since $\odot_N(p)$ is a proper p-induced P-net of N, $\boxdot_{N^{dual}}(p)$ is a proper p-induced T-net of N^{dual} (apply Lemma 1). Since $\boxdot_{N^{dual}}(p)$ is proper and N^{dual} is well-formed, we can apply Lemma 4 to show that $\overline{N^{dual}_{\boxdot(p)}}$ is well-formed. Moreover, $\odot_N(p) = \boxdot_{N^{dual}}(p)$ (see proof of Lemma 4). Hence,

$\overline{N^{dual}_{\boxdot(p)}} = N^{dual} \setminus\!\!\setminus \boxdot_{N^{dual}}(p) = N^{dual} \setminus\!\!\setminus \odot_N(p) = (N \setminus\!\!\setminus \odot_N(p))^{dual} = (\overline{N_{\odot(p)}})^{dual}$ is well-formed. Since $(\overline{N_{\odot(p)}})^{dual}$ is well-formed, also $\overline{N_{\odot(p)}}$ is well-formed (apply Theorem 2 again). Obviously, $\overline{N_{\odot(p)}}$ is free-choice (use Proposition 2(2)).
(3) Both N and $\overline{N_{\odot(p)}}$ are well-formed and free-choice. Hence, both are structurally bounded and covered by P-components. Any P-component of $\overline{N_{\odot(p)}}$ is also a P-component N and initially marked in M because of liveness (use Theorem 5.8 in [9]). Such a P-component is also marked in $M\restriction_{\overline{P}}$. Applying Theorem 5.8 [9] in the other direction proves that $(\overline{N_{\odot(p)}}, M\restriction_{\overline{P}})$ is also live because all P-components are marked. $\square$

5 Approach: Using Induced Subnets For Reduction

Lemmata 4 and 5 show that iteratively removing proper induced T- and P-nets preserves well-formedness, liveness, and boundedness. We first introduce the approach based on *reductions using sequences of proper induced T- and P-nets*. In Section 6, we apply this to properties like lucency and perpetuality.

Definition 13 (Reductions). *Let $N = (P, T, F)$ be a well-formed free-choice net. A reduction of N is a sequence $\gamma = \langle x^1, x^2, \dots, x^n\rangle \in (P \cup T)^*$ such that there exists a sequence of Petri nets denoted $nets_N(\gamma) = \langle N^0, N^1, \dots, N^n\rangle$ where $N^0 = N$, and for any $i \in \{1, \dots, n\}$:*

- *$\boxdot_{N^{i-1}}(x^i)$ is a proper x^i-induced T-net and $N^i = \overline{N^{i-1}{}_{\boxdot(x^i)}}$ if $x^i \in T$.*
- *$\boxdot_{N^{i-1}}(x^i)$ is a proper x^i-induced P-net and $N^i = \overline{N^{i-1}{}_{\odot(x^i)}}$ if $x^i \in P$.*

A reduction $\gamma = \langle x^1, x^2, \dots, x^n\rangle$ is nothing more than a sequence of proper induced T- and P-nets. Figure 2 shows a two-step reduction $\gamma = \langle p7, p1\rangle$ Note that γ uniquely determines $nets_N(\gamma)$. Next, we consider different classes of reductions.

Definition 14 (Complete, T-, and P-Reductions). *Let $N = (P, T, F)$ be a well-formed free-choice net having a reduction $\gamma = \langle x^1, x^2, \dots, x^n\rangle \in (P \cup T)^*$ with the corresponding sequence of Petri nets: $nets_N(\gamma) = \langle N^0, N^1, \dots, N^n\rangle$.*[4]

- *γ is x-preserving if $x \in P \cup T$ is a place/transition in the remaining net N^n.*
- *γ is a complete reduction if N^n is a T-net or a P-net.*
- *γ is a T-reduction if $\{x^1, x^2, \dots, x^n\} \subseteq T$ and N^n is a T-net.*
- *γ is a P-reduction if $\{x^1, x^2, \dots, x^n\} \subseteq P$ and N^n is a P-net.*

The reduction $\gamma_1 = \langle p7, p1\rangle$ illustrated by Figure 2 is a complete P-reduction that is $t4$ preserving. $\gamma_2 = \langle p8, p2\rangle$ is another complete P-reduction and $\gamma_3 = \langle t1\rangle$ and $\gamma_4 = \langle t2\rangle$ are complete T-reductions. Next, we show that such reductions always exist. Moreover, we can preserve any preselected node.

Lemma 6 (Existence of Reductions). *Let $N = (P, T, F)$ be a well-formed free-choice net. N has at least one T-reduction γ_T and at least one P-reduction γ_P. For any node $x \in P \cup T$ there is an x-preserving T-reduction and an x-preserving P-reduction.*

[4] The notions of T-reduction and P-reduction are unrelated to the "Desel rules" for free-choice nets without frozen tokens [8]. We allow for "bigger steps" and can reduce nets with frozen tokens (i.e., there may be an infinite firing sequence starting from a strictly smaller marking).

Proof. Let $N = (P, T, F)$ be a well-formed free-choice net. First, we construct a T-reduction $\gamma_T = \langle t^1, t^2, \ldots, t^n \rangle \in T^*$. If N is a T-net, then $\gamma_T = \langle\ \rangle$ (i.e., $n = 0$). If $N = N^0$ is not a T-net, then there exists a $t^1 \in T$ such that $\boxdot_{N^0}(t^1)$ is proper (Lemma 2). Next, we consider $N^1 = \overline{N^0{}_{\boxdot(t^1)}}$. If N^1 is a T-net, then $\gamma_T = \langle t^1 \rangle$ (i.e., $n = 1$). If N^1 is not a T-net, then there exists a $t^2 \in T$ such that $\boxdot_{N^1}(t^2)$. Etc. This is repeated until we encounter a T-net $N^n = \overline{N^{n-1}{}_{\boxdot(t^n)}}$. We can use the same approach to construct a P-reduction $\gamma_P = \langle p^1, p^2, \ldots, p^m \rangle \in P^*$. If N is a P-net, then $\gamma_P = \langle\ \rangle$. If not, we repeatedly apply Lemma 3 until we find a P-net.

Lemma 2 states that there exist at least two transitions t_1, t_2 such that $\boxdot_N(t_1)$ and $\boxdot_N(t_2)$ are proper. These are disjoint, i.e., $\boxdot_N(t_1) \cap \boxdot_N(t_2) = \emptyset$ (see Proposition 1(8)). Hence, in each step, we can pick an induced T-net not containing a particular node $x \in P \cup T$. The same applies to P-reductions (use Lemma 3 and Proposition 2(8)). $\square$

In Definition 12, we defined update functions for markings that preserve liveness and boundedness. These can be applied in sequence.

Definition 15 (Reduction of Marked Nets). *Let $N = (P, T, F)$ be a well-formed free-choice net having a reduction $\gamma = \langle x^1, x^2, \ldots, x^n \rangle \in (P \cup T)^*$ with the corresponding sequence of nets $nets_N(\gamma) = \langle N^0, N^1, \ldots, N^n \rangle$. In the context of $nets_N(\gamma)$, we denote $N^i = (P^i, T^i, F^i)$ for $i \in \{0, \ldots, n\}$. $mrks_{N,M}(\gamma) = \langle M^0, M^1, \ldots, M^n \rangle$ is such that $M = M^0$ and for any $i \in \{1, \ldots, n\}$:*
- *$M^i = mrk_{\boxdot}(N^{i-1}, x^i, M^{i-1})$ if $x^i \in T$.*
- *$M^i = mrk_{\odot}(N^{i-1}, x^i, M^{i-1})$ if $x^i \in P$.*

Reductions preserved liveness and boundedness, e.g., Figure 2(c) is live and bounded because Figure 2(a) is live and bounded.

Theorem 3 (Reduction Theorem). *Let (N, M) be a live and bounded free-choice net and $\gamma = \langle x^1, x^2, \ldots, x^n \rangle \in (P \cup T)^*$ a reduction of N. Let $nets_N(\gamma) = \langle N^0, N^1, \ldots, N^n \rangle$ and $mrks_{N,M}(\gamma) = \langle M^0, M^1, \ldots, M^n \rangle$ be the corresponding nets and markings. (N^i, M^i) is live and bounded and N^i is well-formed and free-choice for any $i \in \{0, \ldots, n\}$.*

Proof. Let (N, M) be a live and bounded free-choice net, $\gamma = \langle x^1, x^2, \ldots, x^n \rangle$ a reduction, $nets_N(\gamma) = \langle N^0, N^1, \ldots, N^n \rangle$, and $mrks_{N,M}(\gamma) = \langle M^0, M^1, \ldots, M^n \rangle$. We use induction to prove that (N^i, M^i) is live and bounded and N^i is well-formed and free-choice for any $i \in \{0, \ldots, n\}$. If $i = 0$ this holds by definition. Assume $i \geq 1$, (N^{i-1}, M^{i-1}) is live and bounded, and N^{i-1} is well-formed and free-choice (induction hypothesis).

If $x^i \in T$, then $\boxdot_{N^{i-1}}(x^i)$ is proper, $N^i = \overline{N^{i-1}{}_{\boxdot(X^i)}}$, and $M^i = mrk_{\boxdot}(N^{i-1}, x^i, M^{i-1})$. Lemma 4 can be applied to show that (N^i, M^i) is live and bounded and N^i is well-formed and free-choice.

If $x^i \in P$, then $\odot_{N^{i-1}}(x^i)$ is proper, $N^i = \overline{N^{i-1}{}_{\odot(X^i)}}$, and $M^i = mrk_{\odot}(N^{i-1}, x^i, M^{i-1})$. Now, Lemma 5 can be applied to show that (N^i, M^i) is live and bounded and N^i is well-formed and free-choice. This completes the proof by induction. $\square$

The reduction steps are commutative when both are applicable. Consider a reduction $\gamma = \langle x^1, x^2, \ldots, x^n \rangle$ of N, i and j such that $1 \leq i < j \leq n$, and $\gamma' =$

$\langle x^1, x^2, \ldots, x^{i-1}, x^j, x^i, \ldots, x^{j-1}, x^{j+1}, \ldots, x^n \rangle$ (i.e., x^j is moved to the position before x^i). If $x^j \in T$ and $\boxdot_{N^{i-1}}(x^j)$ is proper or $x^j \in P$ and $\odot_{N^{i-1}}(x^j)$ is proper, then γ' is also a reduction of N.

6 Application of Reduction to Prove Perpetuality and Lucency

This section illustrates the usage of reductions. Well-formedness, liveness, and boundedness are preserved "downstream", i.e., these properties are preserved if the net is reduced. For example, N^j is well-formed if N^i is well-formed and $i < j$. We will show that less-common studied properties such as *pc-safeness* and *perpetuality* are also preserved "downstream". Other properties are preserved "upstream", i.e., these properties are preserved if the net is extended. We will use these"upstream" properties to convert results for T-nets or P-nets to free-choice nets (e.g., lucency). First, we introduce three properties that are preserved "downstream".

Definition 16 (Regeneration Transitions). *Let Petri net $N = (P, T, F)$ be a Petri net. Transition $t_r \in T$ is a* regeneration transition *of N if the marked Petri net $(N, [p \in \bullet t_r])$ is live and bounded.*

A *regeneration transition t_r* defines a *regeneration marking $M_r = [p \in \bullet t_r]$*. This can be viewed as a *structural property*: A net is *perpetual* if it has such a marking.

Definition 17 (Perpetual Nets [2]). *Petri net $N = (P, T, F)$ is a* perpetual net *if there exists at least one regeneration transition.*

In a *pc-safe marking* all P-components have precisely one token. Note that a safe marked net does not need to be pc-safe (see, for example, Figure 6 in [2]).

Definition 18 (PC-Safely Marked Nets). *Let Petri net $N = (P, T, F)$ be a Petri net. $M \in \mathcal{B}(P)$ is a* pc-safe *marking of N if for any $X \in PComp(N)$: $M(X \cap P) = 1$, i.e., each P-component contains precisely one token. (N, M) is a pc-safely marked net if M is a pc-safe marking of N.*

In a marked perpetual well-formed free-choice net, regeneration markings can be reached *if and only if* the initial marking is pc-safe.

Lemma 7 (Perpetual Nets Are PC-Safely Marked). *Let $N = (P, T, F)$ be a perpetual well-formed free-choice net with regeneration transition $t_r \in T$. For any marking $M \in \mathcal{B}(P)$: M is pc-safe if and only if $[p \in \bullet t_r] \in R(N, M)$.*

Proof. $M_r = [p \in \bullet t_r]$. (N, M_r) is live and bounded because t_r is a regeneration transition. Take an arbitrary P-component $X \in PComp(N)$. $M_r(X \cap P) \neq 0$, because, otherwise, the transitions in $X \cap T$ would be dead contradicting liveness. $M_r(X \cap P) \not> 1$, because this implies that one of the input places of t_r has at least two tokens. Hence, M_r is pc-safe and all P-components contain precisely one input place of t_r. If $M_r \in R(N, M)$, then M needs to be pc-safe (the number of tokens in a P-component cannot change). Remains to show that M_r can be reached from M if M is pc-safe. (N, M) is live if M is pc-safe (use Theorem 5.8 in [9]). Hence, t_r can be enabled, proving that M_r is indeed reachable. $\square$

Next, we show that the properties just defined are preserved "downstream" *for any reduction* (i.e., also for mixtures of place- and transition-induced subsets).

Theorem 4 (Invariant Downstream Properties). *Let $N = (P, T, F)$ be a well-formed free-choice net having a reduction $\gamma = \langle x^1, x^2, \ldots, x^n \rangle$ with the corresponding sequence of nets $nets_N(\gamma) = \langle N^0, N^1, \ldots, N^n \rangle$.*

(1) If $t_r \in T$ is a regeneration transition of N (i.e., $(N, [p \in \bullet t_r])$ is live and bounded) and γ is t_r-preserving, then t_r is a regeneration transition of all nets in $nets_N(\gamma)$ (i.e., $(N^i, [p \in \bullet t_r])$ is live and bounded for any $i \in \{0, \ldots, n\}$).[5]

(2) If (N, M) is pc-safe, then all markings in $mrks_{N,M}(\gamma)$ are pc-safe.

(3) If N is perpetual, then all nets in $nets_N(\gamma)$ are perpetual.

Proof. Let $N = (P, T, F)$ be a well-formed free-choice net having a reduction $\gamma = \langle x^1, x^2, \ldots, x^n \rangle$ and $nets_N(\gamma) = \langle N^0, N^1, \ldots, N^n \rangle$.

(1) Assume $(N, [p \in \bullet t_r])$ is live and bounded and γ is t_r-preserving. We prove that $(N^i, [p \in \bullet t_r])$ is live and bounded for any $i \in \{0, \ldots, n\}$ using induction. If $i = 0$, this holds by definition ($N^0 = N$). Assume $i \geq 1$ and $(N^{i-1}, [p \in \bullet t_r])$ is live and bounded. If $x^i \in T$, then $\boxdot_{N^{i-1}}(x^i)$ is proper, $N^i = \overline{N^{i-1}{}_{\boxdot(X^i)}}$, and $M^i = mrk_{\boxdot}(N^{i-1}, x^i, [p \in \bullet t_r]) = [p \in \bullet t_r]$, because t_r and $\bullet t_r$ are outside $\boxdot_{N^{i-1}}(x^i)$ (γ is t_r-preserving).[5] We can apply Theorem 3 to show that $(N^i, [p \in \bullet t_r])$ is live and bounded. If $x^i \in P$, then $\odot_{N^{i-1}}(x^i)$ is proper, $N^i = \overline{N^{i-1}{}_{\odot(X^i)}}$, and $M^i = mrk_{\odot}(N^{i-1}, x^i, [p \in \bullet t_r]) = [p \in \bullet t_r]\restriction_{P^i}$. t_r is not removed because γ is t_r-preserving. Hence, also at least one input place of t_r remains. Therefore, $M^i = [p \in \bullet t_r]$ (note that $\bullet t_r$ may have been changed[5]) and $(N^i, [p \in \bullet t_r])$ is live and bounded (apply again Theorem 3). Hence, t_r is a regeneration transition of all nets in $nets_N(\gamma)$.

(2) Assume (N, M) is pc-safe and $mrks_{N,M}(\gamma) = \langle M^0, M^1, \ldots, M^n \rangle$. Again we use induction and prove that (N^i, M^i) is pc-safe for any $i \in \{0, \ldots, n\}$. If $i = 0$, this holds by definition ($(N^0, M^0) = (N, M)$ is pc-safe). Assume $i \geq 1$ and (N^{i-1}, M^{i-1}) is pc-safe (induction hypothesis). We need to show that (N^i, M^i) is pc-safe. Take an arbitrary P-component $X \in PComp(N^i)$, we need to show that $M^i(X \cap P) = 1$.

- If $x^i \in P$, then $PComp(N^i) \subseteq PComp(N^{i-1})$ because for the remaining places the context did not change. Also the marking of the remaining places does not change, because $M^i = mrk_{\odot}(N^{i-1}, x^i, M^{i-1}) = M^i\restriction_{P^i}$. Hence, $M^i(X \cap P) = 1$.
- If $x^i \in T$, but $X \in PComp(N^{i-1})$, then nothing changed and $M^i(X \cap P) = 1$ (note that in a P-component all surrounding transitions are included, hence the marking of the places in X and their context, i.e., pre- and post-sets, did not change).
- Assume $x^i \in T$ and $X \notin PComp(N^{i-1})$. Let $P_X = X \cap P$ be the places in the P-component X (these are outside the x^i-induced T-net) and $T_X = \boxdot_{N^{i-1}}(x^i) \cap T$ the transitions in the x^i-induced T-net. $F_{in} = F^{i-1} \cap (P_X \times T_X)$ are the ingoing arcs and $F_{out} = F^{i-1} \cap (T_X \times P_X)$ are the outgoing arcs. Both sets need to have precisely one element, i.e., $F_{in} = \{(p_{in}, t_{in})\}$ and $F_{out} = \{(t_{out}, p_{out})\}$, and $t_{in} = x^i$. One of these two sets of arcs is non-empty because P_X must contain at least one place that was connected to a transition T_X and if one is non-empty the other one is also non-empty. Proposition 1(7) implies that $t_{in} = x^i$

[5] Note that $\bullet t_r = \{p \mid (p, t_r) \in F^i\}$ depends on the net considered (here N^i).

and P_X cannot hold two input places of t_{in} because of Proposition 1(6). p_{in} is the unique input place in X. F_{out} cannot have multiple elements because N^{i-1} is well-formed and therefore structurally bounded. Consider now an elementary path $\rho = \langle t_{in}, p_1, \ldots, p_n, t_{out} \rangle \in (\boxdot_{N^{i-1}}(x^i))^*$. Such a path must exist and the places are non-branching. $Y = X \cup \{x \in \rho\}$ is a P-component because Y is strongly connected, all places in Y are non-branching, and all input and output transitions are included. Hence, $Y \in PComp(N^{i-1})$ and $M^{i-1}(Y) = 1$ because (N^{i-1}, M^{i-1}) is pc-safe. Moreover, $M^{i-1}(Y) = M^i(X)$ (pushing out the tokens does not change the total number of tokens, and X must be marked in M^i). Hence, $M^i(X) = 1$.

(3) Assume that N is perpetual. To show that all nets in $nets_N(\gamma)$ are perpetual, the same approach can be used as in (1). The only difference is that there is not a fixed regeneration transition t_r that is preserved. Assume that $(N^{i-1}, [p \in \bullet t_r])$ is live and bounded. We need to show that there is a t'_r such that $(N^i, [p \in \bullet t'_r])$ is live and bounded. If $t_r \in N^i$ (i.e., the regeneration transition is outside the x^i-induced subset), then $t_r = t'_r$ and this transition remains a regeneration transition (as shown in (1)). If $t_r \notin N^i$, then we need to consider two cases:

- If $x^i \in P$ and $t_r \notin N^i$, then we find a contradiction, because t_r, like any regeneration transition, should be in all P-components of N^{i-1}. This is impossible, because this implies $M^i = mrk_{\odot}(N^{i-1}, x^i, [p \in \bullet t_r]) = [\,]$.
- If $x^i \in T$ and $t_r \notin N^i$, then pick $t'_r \in N^i$ such that $\bullet t'_r = \bullet x^i$. Proposition 1(6) shows that such a transition exists. t'_r is live in $(N^{i-1}, [p \in \bullet t_r])$. Consider a reachable marking enabling t'_r and then "push out" as many tokens as possible using the same approach as in Proposition 4. Let M be the marking where t'_r and x^i are enabled and all other transitions in $\boxdot_{N^{i-1}}(x^i)$ are not. From M we must be able to enable the regeneration transition t_r by only firing transitions in $\boxdot_{N^{i-1}}(x^i)$ (other transitions can only influence the subnet through x^i). Therefore, all other places $P^i \setminus \bullet t'_r$ must be empty in M, showing that $(N^i, [p \in \bullet t'_r])$ is live and bounded.

Hence, using a similar approach as in (1) we showed that N^i is perpetual for any i. $\square$

Next, we consider lucency, first defined in [2]. We are often interested in processes where the set of enabled actions uniquely defines the state, e.g., in the context of process mining or user-interface design [2, 4]. In terms of Petri nets, this means that there cannot be two reachable marking enabling the same set of transitions.

Definition 19 (Lucency [2]). *Petri net $N = (P, T, F)$ is* lucent *if each pc-safe marking enables a unique set of transitions, i.e., for any two pc-safe markings M_1 and M_2: if $en(N, M_1) = en(N, M_2)$, then $M_1 = M_2$.*

After showing that well-formedness, liveness, boundedness, pc-safeness, and perpetuality are preserved "downstream", we show that lucency is preserved by traversing the reduction in "upstream" direction. This is non-trivial because even live and pc-safe free-choice nets may be non-lucent [2]. Therefore, we first present some results for perpetual nets, before using a T-reduction to prove that perpetuality implies lucency.

Lemma 8 (Identical Token Counts On Related Paths). *Let $N = (P, T, F)$ be a perpetual well-formed free-choice net. Let $M \in \mathcal{B}(P)$ be a pc-safe marking of N,*

$t_b, t_e \in T$ be two transitions, and $\rho_1 = \langle t_b, p_1^1, t_1^1, p_1^2, t_1^2, \ldots, p_1^m, t_e \rangle$ and $\rho_2 = \langle t_b, p_2^1, t_2^1, p_2^2, t_2^2, \ldots, p_2^n, t_e \rangle$ be two elementary paths leading from t_b and t_e covering places $P_1 = \{p_1^1, p_1^2, \ldots, p_1^m\}$ and $P_2 = \{p_2^1, p_2^2, \ldots, p_2^n\}$ such that for any $p \in P_1 \cup P_2$: $|\bullet p| = |p \bullet| = 1$. $M(P_1) = M(P_2)$, i.e., the number of tokens on both paths is identical.

Proof. The number of tokens on both elementary paths ρ_1 and ρ_2 is only changed by t_b and t_e. All other transitions are either not connected to any place $p \in P_1 \cup P_2$ or move a token to the next place on the path. t_b adds a token to both paths and t_e removes a token from both paths. Hence, the difference $M'(P_1) - M'(P_2)$ remains constant for any $M' \in R(N, M)$.

Assume that $M(P_1) \neq M(P_2)$. This implies that $M'(P_1) \neq M'(P_2)$ for any $M' \in R(N, M)$. This includes $M_r(P_1) \neq M_r(P_2)$ for the regeneration marking $M_r = [p \in \bullet t_r]$ based on a regeneration transition t_r. Due to Lemma 7, M_r is pc-safe and can be reached from any pc-safe marking. Without loss of generality, we may assume $M_r(P_1) > M_r(P_2)$ (we can swap P_1 and P_2), i.e., there is a place $p_r \in \bullet t_r \cap P_1$ marked in the regeneration marking M_r. t_r cannot have two input places from P_1 because all places in P_1 have one output transition which is unique. Hence, $M_r(P_1) = 1$ implying that $M_r(P_2) = 0$. Hence, $M'(P_1) = M'(P_2) + 1$ for any $M' \in R(N, M)$. Because N is perpetual, all transitions are live, including t_b. After t_b fires, there is at least one token in P_2 until t_e fires. This implies that there are at least two tokens in P_1 until t_e fires. However, t_e cannot be reached without executing first t_r, but when executing t_r, p_r must be the only marked place in P_1 containing precisely one token leading to a contradiction. Hence, $M(P_1) = M(P_2)$. □

We introduce *conflict-pairs* as "witnesses" of non-lucency. If a T-net is not lucent, then it must have a conflict-pair (Proposition 5).

Definition 20 (Conflict-Pair). *Let N be a Petri net. (M_1, M_2) is called a* conflict-pair *for N if (N, M_1) and (N, M_2) are pc-safely marked, $en(N, M_1) \cap en(N, M_2) = \emptyset$ (no transition is enabled in both markings), for all $t \in en(N, M_1)$: $M_2(\bullet t) \geq 1$, and for all $t \in en(N, M_2)$: $M_1(\bullet t) \geq 1$.*

Proposition 5 (Absence of Conflict-Pairs in T-nets Implies Lucency). *Let N be a perpetual well-formed T-net. If N is not lucent, then N has conflict-pairs.*

Proof. Assume N is not lucent, i.e., there are two pc-safe markings M_1 and M_2 such that $en(N, M_1) = en(N, M_2)$ and $M_1 \neq M_2$. Tokens in M_1 but not in M_2 are represented by ① and tokens in M_2 but not in M_1 are represented by ②. These ① and ② tokens can be viewed as "disagreement tokens", i.e., M_1 and M_2 disagree on the marking of the corresponding place. Tokens in both markings are denoted by • and are called "agreement tokens". We now synchronously modify the markings M_1 and M_2 by firing only transitions using "agreement tokens" (•) and not consuming any of the "disagreement tokens" (① and ②). Because N is perpetual, there is regeneration transition $t_r \in T$. Since M_1 and M_2 are pc-safe, $M_r = [p \in \bullet t_r]$ can be reached by both. Consider a shortest firing sequence σ from M_1 to M_r: $(N, M_1)[\sigma\rangle(N, M_r)$. Try to execute the sequence without consuming any of the ① tokens. Transitions that need to consume "disagreement tokens" or that are disabled can be skipped. However, per cluster transitions are executed in the same order as in σ (note that if one transition in the cluster is enabled, all are). This is repeated until there are no transitions enabled using only "agreement tokens". This process can be formalized by considering the partially-ordered run corresponding to the firing sequence σ from M_1 to M_r. Remove all transition consuming "disagreement tokens" from the partially-ordered run and execute the run as far as possible. Let M_1' be the resulting marking and σ' the partial sequence such that $(N, M_1)[\sigma'\rangle(N, M_1')$. σ' can also be executed starting from M_2 since only agreement tokens are used. Let M_2' be such that $(N, M_2)[\sigma'\rangle(N, M_2')$. Also in M_2' all enabled transitions need to consume "disagreement tokens" (i.e., ② tokens).

(N, M_1') and (N, M_2') are pc-safely marked, $en(N, M_1') \cap en(N, M_2') = \emptyset$ (otherwise a transition using agreement tokens is enabled), for all $t \in en(N, M_1)$: $M_2(\bullet t) \geq 1$, and for all $t \in en(N, M_2)$: $M_1(\bullet t) \geq 1$ (because we did not produce new disagreement tokens, no transition is enabled based on disagreement tokens only). □

The goal is to show that perpetual free-choice nets are lucent. To do this, we construct a T-reduction where perpetuality is preserved "downstream" and lucency is preserved "upstream". For the "upstream reasoning" we start from a T-net. Hence, we first show that any perpetual well-formed T-net is lucent (using conflict-pairs as witnesses of non-lucency and Lemma 8 to show that such witnesses cannot exist).

Theorem 5 (Perpetual T-nets Have No Conflict-Pairs). *Let $N = (P, T, F)$ be a perpetual well-formed T-net. N does not have any conflict-pairs.*

Proof. Let $N = (P, T, F)$ be a perpetual well-formed T-net with regeneration transition $t_r \in T$. $M_r = [p \in \bullet t_r]$ is a regeneration marking (i.e., $(N, [p \in \bullet t_r])$ is live and bounded). (N, M_r) is also pc-safe (Lemma 7). Assume N has a conflict-pair (M_1, M_2), i.e., (N, M_1) and (N, M_2) are pc-safely marked, $en(N, M_1) \cap en(N, M_2) = \emptyset$, for all $t \in en(N, M_1)$: $M_2(\bullet t) \geq 1$, and for all $t \in en(N, M_2)$: $M_1(\bullet t) \geq 1$. Note that for any $X \in PComp(N)$: $M_1(X) = M_2(X) = M_r(X) = 1$. Each circuit is a P-component of N (and vice versa) and contains precisely one token in any marking considered. This implies that each circuit includes t_r. $T_D = \{t \in T \setminus \{t_r\} \mid \exists_{p \in \bullet t} M_1(p) \neq M_2(p)\}$ are all transitions that disagree on at least one of the input places (excluding t_r). Note that $T_D \neq \emptyset$ (M_1 and M_2 disagree on at least one P-component, yielding two disagreeing transitions). Pick a disagreeing transition t_D such that there is no other disagreeing transition on a path from t_r to t_D. This is possible because each circuit includes t_r, i.e., there are no cycles not involving the regeneration transition. Without loss of generality we may assume that there is a place $p_D \in \bullet t_D$ such that $M_1(p_D) = 1$ and $M_2(p_D) = 0$. t_D must have at least one other input place p_A that is not just marked in M_1, i.e., $M_1(p_A) \leq M_2(p_A)$ (otherwise (M_1, M_2) is not a conflict-pair).

Now we can apply Lemma 8 using the elementary paths $\rho_1 = \langle t_b, p_1^1, t_1^1, p_1^2, t_1^2, \ldots, p_1^m, t_e\rangle$ and $\rho_2 = \langle t_b, p_2^1, t_2^1, p_2^2, t_2^2, \ldots, p_2^n, t_e\rangle$ with $t_b = t_r$, $t_e = t_D$, $p_1^m = p_D$, $p_2^n = p_A$, and $|\bullet p| = |p \bullet| = 1$ for any $p \in P_1 \cup P_2$. Hence, Lemma 8 implies that $M_1(P_1) = M_1(P_2)$ and $M_2(P_1) = M_2(P_2)$.

We picked t_D such that there is no other disagreeing transition on a path from t_r to t_D. Hence, M_1 and M_2 agree on $P_1 \setminus \{p_D\} = \{p_1^1, p_1^2, \ldots, p_1^{m-1}\}$ and $P_2 \setminus \{p_A\} = \{p_2^1, p_2^2, \ldots, p_2^{n-1}\}$, i.e., $M_1(p) = M_2(p)$ for all $p \in (P_1 \cup P_2) \setminus \{p_D, p_A\}$. $M_1(p_D) > M_2(p_D)$ and $M_1(p_A) \leq M_2(p_A)$. Therefore, $M_1(P_1) > M_2(P_1)$ and $M_1(P_2) \leq$

$M_2(P_2)$. Combined with $M_1(P_1) = M_1(P_2)$ and $M_2(P_1) = M_2(P_2)$ this leads to a contradiction. Hence, (M_1, M_2) cannot be a conflict-pair of N. $\square$

Corollary 1 (Perpetual T-nets Are Lucent). *Let $N = (P, T, F)$ be a perpetual well-formed T-net. N is lucent.*

Proof. Follows directly from Proposition 5 and Theorem 5. $\square$

Starting from a perpetual well-formed free-choice net and a T-reduction, we show that lucency is preserved in the "upstream" direction. We first prove that the absence of conflict-pairs is preserved "upstream" and use Theorem 5 as the base case. To simplify the proof, we assume that a particular regeneration transition t_r is preserved, but this is not essential and this requirement could be dropped (see last part of Theorem 4).

Theorem 6 (T-Reduction Showing Absence of Conflict-Pairs). *Let N be a perpetual well-formed free-choice net having a regeneration transition $t_r \in T$ and a T-reduction $\gamma_T = \langle t^1, t^2, \ldots, t^n \rangle$ that is t_r preserving. None of the Petri-nets in $nets_N(\gamma_T) = \langle N^0, N^1, \ldots, N^n \rangle$ has conflict-pairs.*

Proof. Assume that N is a perpetual well-formed free-choice net with regeneration transition $t_r \in T$ and the T-reduction $\gamma_T = \langle t^1, t^2, \ldots, t^n \rangle$ is t_r preserving (it is always possible to create such T-reduction). $nets_N(\gamma_T) = \langle N^0, N^1, \ldots, N^n \rangle$.

Using Theorem 4 we know that $N^i = (P^i, T^i, F^i)$ is perpetual for any $i \in \{0, \ldots, n\}$. We need to show that N^i has no conflict-pairs. We use induction in the reverse direction starting with $i = n$. Base case: N^n is a T-net and has no conflict-pairs (Theorem 5). Induction step: We need to show that if N^i has no conflict-pairs, N^{i-1} has no conflict-pairs. This is the same as showing that if N^{i-1} has conflict-pairs, N^i also has conflict-pairs. To simplify notation we introduce the shorthands: $N = N^{i-1} = (P, T, F)$, $N' = N^i = (P', T', F')$ and $t = t^i$, i.e., $\boxdot_N(t)$ is a proper t-induced T-net and $N' = \overline{N_{\boxdot(t)}}$.

Let (M_1, M_2) be a conflict-pair for N, i.e., (N, M_1) and (N, M_2) are pc-safely marked, $en(N, M_1) \cap en(N, M_2) = \emptyset$, for all $t' \in en(N, M_1)$: $M_2(\bullet t') \geq 1$, and for all $t' \in en(N, M_2)$: $M_1(\bullet t') \geq 1$. Based on (M_1, M_2) we construct (M_1', M_2') with $M_1' = mrk_{\boxdot}(N, t, M_1)$ and $M_2' = mrk_{\boxdot}(N, t, M_2)$. We need to show that (M_1', M_2') is a conflict-pair. (N', M_1') and (N', M_2') are pc-safely marked (use Theorem 4). The remaining requirements in Definition 20 are shown by case distinction.

If $M_1 \upharpoonright \boxdot_N(t) = [\,]$ and $M_2 \upharpoonright \boxdot_N(t) = [\,]$, then $M_1' = M_1$, $M_2' = M_2$, and (M_1', M_2') is indeed a conflict-pair for N' (it is easy to verify that the requirements in Definition 20 still hold).

If $M_1 \upharpoonright \boxdot_N(t) \neq [\,]$ or $M_2 \upharpoonright \boxdot_N(t) \neq [\,]$, then at least one transition in $\boxdot_N(t)$ has a token in its input place. Let $T_D = \{t' \in (T \cap \boxdot_N(t)) \setminus \{t\} \mid M_1(\bullet t') + M_2(\bullet t') \geq 1\}$ (i.e., all transitions have a marked input place in one of the two markings). Pick a transition $t_D \in T_D$ such that there is no other T_D transition on a path from t to t_D. This is possible because there are no cycles inside $\boxdot_N(t)$ and there is a path from t to any node in $\boxdot_N(t)$. If there would be a cycle, then the regeneration transition t_r needs to be in $\boxdot_N(t)$, which is not the case because t_r is preserved (actually, t_r is a regeneration transition of N'). See also Theorem 5, which uses similar reasoning.

One of the input places of t_D is marked in M_1 or M_2. Since (M_1, M_2) is a conflict-pair for N, t_D cannot be enabled in both. Hence, for at least one of the two markings M_1 or M_2, we can find two input places that "disagree" (check all cases using Definition 20). Without loss of generality, let us assume that $p_m, p_u \in \bullet t_D$, $p_m \in M_1$, and $p_u \notin M_1$, i.e., the input places p_m and p_u of t_D and marking are chosen such that p_m is marked and p_u is not. Moreover, all places on a path from t to these places are empty. Just like in Theorem 5 and Lemma 8, we create two elementary paths: $\rho_1 = \langle t_b, p_1^1, t_1^1, p_1^2, t_1^2, \ldots, p_1^m, t_e\rangle$ and $\rho_2 = \langle t_b, p_2^1, t_2^1, p_2^2, t_2^2, \ldots, p_2^n, t_e\rangle$, now with $t_b = t$, $t_e = t_D$, $p_1^m = p_m$, and $p_2^n = p_u$. All places on these two paths are empty in M_1 except $p_1^m = p_m$. This leads to a contradiction using Lemma 8, which states that the number of tokens on both paths should be identical. Hence, $M_1\restriction \boxdot_N(t) = M_2\restriction \boxdot_N(t) = [\,]$, $M_1' = M_1$, $M_2' = M_2$, and (M_1', M_2') is indeed a conflict-pair for N'. $\square$

Corollary 2 (Perpetual Free-Choice Nets Are Lucent). *All perpetual well-formed free-choice nets are lucent.*

Proof. Follows directly from Proposition 5 and Theorem 6. $\square$

Corollary 2 corresponds to Theorem 3 in [2]. As pointed out earlier by the author in e.g. [3], the initial proof of Theorem 3 in [2] was incomplete and a repaired proof was provided [3]. When repairing the proof, the author discovered that the result also holds for non-well-formed perpetual free-choice nets. A detailed proof is given in [5]. This more general result uses a completely different approach and does not build upon existing results for well-formed free-choice nets.

Note that for any reduction $\gamma = \langle t^1, t^2, \ldots, t^n\rangle$ of a perpetual well-formed free-choice net all nets in $nets_N(\gamma) = \langle N^0, N^1, \ldots, N^n\rangle$ are lucent and free of conflict-pairs. Hence, it is also possible to provide alternative versions of Theorem 6 using a P-reduction and the fact that lucency trivially holds for perpetual P-nets.

The approach presented in this section can also be used to prove the so-called *blocking theorem* [12, 15] which states that every cluster in a bounded and live free-choice system has a unique marking enabling the cluster. This can be seen as lucency for individual transitions without requiring perpetuality. To prove the blocking theorem, we first show that blocking markings exist by moving tokens towards the selected cluster (this is possible due to the free-choice properly). Moreover, the uniqueness of blocking markings is preserved "upstream" and holds for T-nets (similar to Theorem 5, but using the fact that in blocking markings all transitions outside the selected cluster have empty input places). A detailed proof is straightforward, but omitted for space reasons.

7 Conclusion

This paper proposed *reductions* based on sequences of proper t-induced T-nets and p-induced P-nets. Such a reduction can be used to transform any free-choice net into a T-net or P-net. Given an arbitrary reduction γ, properties are preserved "downstream" (e.g., well-formedness, liveness, pc-safety, and perpetuality) and "upstream" (e.g., lucency and the absence of conflict-pairs, assuming perpetuality). Using the framework, we could reconfirm classical and more recent results related to lucency and perpetuality in a systematic manner. The framework is general and can be used for other properties, e.g., it becomes straightforward to prove the well-known blocking theorem [12, 15] using a T-reduction.

The theoretical work presented was driven by challenges in the field of process mining. Process discovery techniques greatly benefit from additional assumptions such as lucency and perpetuality [4]. Moreover, we want to extend our work on *interactive* and *incremental* process mining using t-induced T-nets and p-induced P-nets. An obvious limitation of the current framework is that well-formedness is preserved "downstream" but not "upstream". However, the approach can be adapted to work in the reverse direction (using P-covers and T-covers).

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